A Stable Magnetic Background in SU(2) QCD

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Motivated by the instability of the Savvidy-Nielsen-Olesen (SNO) vacuum we make a systematic search for a stable magnetic background in pure $SU(2)$ QCD. It is shown that Wu-Yang monopole-antimonopole pair is unstable under vacuum fluctuations. However, it is shown that a pair of axially symmetric monopole-antimonopole string configuration is stable, provided the distance between the two strings is small enough (less than a critical value). The existence of a stable monopole-antimonopole string background strongly supports that a magnetic condensation of monopole-antimonopole pairs can indeed generate a dynamical symmetry breaking, and thus a desired magnetic confinement of color, in QCD.

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I. INTRODUCTION

One of the most outstanding problems in theoretical physics is the confinement problem in QCD. It has long been argued that the monopole condensation could explain the confinement of color through the dual Meissner effect [1, 2]. Indeed, if one assumes the monopole condensation, one could easily argue that the ensuing dual Meissner effect guarantees the confinement [3]. There have been many attempts to prove this scenario in QCD [4, 5]. Unfortunately the earlier attempts has failed to establish the desired magnetic condensation, because the magnetic background, known as the Savvidy-Nielsen-Olesen (SNO) vacuum, is not stable. In fact the effective action of QCD obtained with the SNO vaccum develops an imaginary part, which implies that the SNO vacuum is unstable [6, 7]. This instability of the magnetic condensation has been widely accepted and never been convincingly revoked.

In retrospect there are many reasons why the earlier attempts have not been so successful. First, the calculation of the effective action has involved tachyons which violates the causality, a fundamental principle in quantum field theory. Indeed it is well-known that the imaginary part of the effective action originates from the tachyonic contribution. This tells that the causality principle may have been compromised in the calculation of the effective action. Secondly, the calculation of the effective action was not gauge independent. In fact the SNO background itself was not gauge invariant [6, 7]. And obviously any background which is not gauge invariant can not possibly become a stable vacuum. From these points of view it is really not suprising that the SNO vacuum turns out to be unstable. There have been attempts to cure this defect of the SNO vacuum and prove the magnetic condensation with a gauge invariant background, but unfortunately these attempts have not been very successful [5, 6, 7].

Recently, however, this instability of the SNO vacuum has been studied more carefully. It has been shown that, if one uses a proper infra-red regularization which respects causality, the imaginary part in the effective action disappears [8, 9, 10]. Furthermore, it has been argued that the imaginary part of the effective action disappears if one imposes the gauge invariance to the SNO vacuum correctly. Indeed the calculation of the effective action based on color reflection invariance shows that the effective action has no imaginary part [11]. This implies that a “gauge-invariant” SNO vacuum can be qualified as a stable vacuum of QCD.

Nevertheless, the meaning of the “gauge-invariant” SNO background has not been fully understood so far. In

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particular, an explicit example of stable magnetic background has never been constructed. To understand the complexity of the problem, notice that the color charge in QCD can uniquely be defined only after one selects the color direction. But the color direction in QCD is gauge dependent. So it is a non-trivial matter to construct a "gauge-invariant" magnetic vacuum.

There is an intuitive way to understand why the SNO background is unstable. The SNO vacuum is a constant color magnetic field pointed to a fixed direction in the color $SU(2)$ space. Unfortunately this vacuum configuration is not gauge invariant. To see this consider a second magnetic vacuum which has the opposite color direction. Clearly the two vacua are distinct because they have opposite color. Nevertheless they are gauge equivalent, because one can always rotate the first vacuum to the second one with a gauge transformation [11]. This means that neither the first nor the second vacuum is gauge invariant. Only the gauge invariant combination of the two vacua becomes gauge invariant, and thus could become a physical vacuum. This is why the SNO vacuum must be unstable.

There is another example which has exactly the same instability problem. Consider the Wu-Yang monopole [12, 13]. In spite of its topological origin it is well known that the monopole is unstable [14]. This is because it is not gauge invariant. Here again we have the anti-monopole which is gauge equivalent to the monopole, and only a gauge invariant combination of the monopole and anti-monopole can exist as a physical (gauge invariant) object, just as a gauge invariant combination of quark and anti-quark can exist as physical. This tells that a physical monopole condensation should not be a simple monopole condensation, but a condensation of gauge invariant combination of monopole and anti-monopole pairs.

The above heuristic argument tells that only a gauge invariant SNO vacuum, if at all, has a chance to become a physical vacuum of QCD [11]. *The purpose of this paper is to search for a stable magnetic background in $SU(2)$ QCD. We analyze the stability of two classical magnetic backgrounds, a pair of axially symmetric monopole-antimonopole strings and a pair of magnetic vortex-antivortex strings, and show that the pair of monopole-antimonopole string configuration becomes stable provided the distance between two strings is small enough.* As far as we understand, the pair of axially symmetric monopole-antimonopole strings constitutes a first explicit example of a stable magnetic background in QCD. More importantly the result can serve as a strong argument that a gauge invariant monopole-antimonopole condensation can provide a stable vacuum in QCD. This reinforces the claim that the "gauge-invariant" SNO vacuum could generates a desired dynamical symmetry breaking which could confine the color in QCD.

The paper is organized as follows. In Section II we review the geometric structure of the connection space in QCD, and discuss how one can obtain a gauge independent separation of a classical background from the quantum fluctuation. In Section III we review the SNO effective action of QCD to clarify the origin of the instability of the SNO vacuum. In Section IV we discuss a gauge invariant calculation of the effective action, and show how the gauge invariance can cure the instability of the SNO vacuum. In Section V we analyze the stability of the classical Wu-Yang monopole and anti-monopole pair, and show that the configuration is unstable. In Section VI we consider an axially symmetric monopole string (a two-dimensional classical monopole configuration), and show that it is unstable under the quantum fluctuation. In Section VII we consider a pair of axially symmetric monopole and anti-monopole strings, and show that the configuration is stable if the distance between the two strings is small enough. In Section VIII we study the stability of an axially symmetric magnetic vortex-antivortex pair, and show that the vortex pair is unstable. Finally in Section IX we discuss the physical significance of our result.

II. GAUGE INDEPENDENT DECOMPOSITION OF NON-ABELIAN GAUGE POTENTIAL

One of the conceptional problems in non-Abelian gauge theory is how to define the color. It is well known that the conserved color charge is gauge dependent. Indeed the gauge-dependence of the conserved color is so severe that one can always choose a gauge in which the color charge becomes identically zero. This means that, to discuss the confinement of color, one must know how to define the color in a gauge independent way. Consider $SU(2)$ QCD for simplicity. A natural way to define the color is to introduce an isotriplet (a unit vector field in color space) $\hat{n}$ which selects the color direction at each space-time point, and to decompose the gauge potential into the restricted potential $\hat{A}_\mu$ which leaves $\hat{n}$ invariant and the valence potential $\vec{X}_\mu$ which forms a covariant vector field [2, 3],

$$\vec{A}_\mu = A_\mu \hat{n} - \frac{1}{g}\hat{n}\times\partial_\mu\hat{n} + \vec{X}_\mu = \hat{A}_\mu + \vec{X}_\mu,$$
$$(\hat{n}^2 = 1, \quad \hat{n}\cdot\vec{X}_\mu = 0), \qquad (1)$$

where $A_\mu = \hat{n}\cdot\vec{A}_\mu$ is the "electric" potential. Clearly this way of selecting the color direction is gauge independent, because $\hat{n}$ is chosen to be gauge covariant.

Notice that the restricted potential is precisely the connection which leaves $\hat{n}$ invariant under the parallel transport,

$$\hat{D}_\mu\hat{n} = \partial_\mu\hat{n} + g\hat{A}_\mu\times\hat{n} = 0. \qquad (2)$$

Under the infinitesimal gauge transformation

$$\delta\hat{n} = -\vec{\alpha}\times\hat{n}\,,\quad \delta\vec{A}_\mu = \frac{1}{g}D_\mu\vec{\alpha}, \tag{3}$$

one has

$$\delta A_\mu = \frac{1}{g}\hat{n}\cdot\partial_\mu\vec{\alpha},\quad \delta\hat{A}_\mu = \frac{1}{g}\hat{D}_\mu\vec{\alpha},$$
$$\delta\vec{X}_\mu = -\vec{\alpha}\times\vec{X}_\mu. \tag{4}$$

This shows that $\hat{A}_\mu$ by itself describes an $SU(2)$ connection which enjoys the full $SU(2)$ gauge degrees of freedom. Furthermore $\vec{X}_\mu$ transforms covariantly under the gauge transformation. Most importantly, the decomposition is gauge-independent. Once the color direction $\hat{n}$ is selected, the decomposition follows automatically, independent of the choice of a gauge.

The restricted potential $\hat{A}_\mu$ actually has a dual structure. Indeed the field strength made of the restricted potential is decomposed as

$$\hat{F}_{\mu\nu} = (F_{\mu\nu}+H_{\mu\nu})\hat{n},$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$
$$H_{\mu\nu} = -\frac{1}{g}\hat{n}\cdot(\partial_\mu\hat{n}\times\partial_\nu\hat{n}) = \partial_\mu\tilde{C}_\nu - \partial_\nu\tilde{C}_\mu, \tag{5}$$

where $\tilde{C}_\mu$ is the "magnetic" potential [2, 3]. Notice that we can always introduce the magnetic potential (at least locally section-wise), because $H_{\mu\nu}$ forms a closed two-form

$$\partial_\mu\tilde{H}_{\mu\nu} = 0 \qquad (\tilde{H}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}H_{\rho\sigma}). \tag{6}$$

This allows us to identify the non-Abelian magnetic potential by

$$\vec{C}_\mu = -\frac{1}{g}\hat{n}\times\partial_\mu\hat{n}. \tag{7}$$

Indeed with $\hat{n}=\hat{r}$ the magnetic potential describes the well-known Wu-Yang minopole [12, 13].

With the decomposition (1) one has

$$\vec{F}_{\mu\nu} \;=\; \hat{F}_{\mu\nu} + \hat{D}_\mu\vec{X}_\nu - \hat{D}_\nu\vec{X}_\mu + g\vec{X}_\mu\times\vec{X}_\nu, \tag{8}$$

so that the Lagrangian can be written as follows

$$\mathcal{L} = -\frac{1}{4}\hat{F}_{\mu\nu}^2 - \frac{1}{4}(\hat{D}_\mu\vec{X}_\nu - \hat{D}_\nu\vec{X}_\mu)^2$$
$$-\frac{g}{2}\hat{F}_{\mu\nu}\cdot(\vec{X}_\mu\times\vec{X}_\nu) - \frac{g^2}{4}(\vec{X}_\mu\times\vec{X}_\nu)^2. \tag{9}$$

This shows that QCD is a restricted gauge theory which has a gauge covariant valence gluon as a colored source.

The decomposition (1), which has recently been referred to as the "Cho decomposition" or "Cho-Faddeev-Niemi decomposition" [15, 16, 17, 18], was first introduced long time ago in an attempt to demonstrate the monopole condensation in QCD [2, 3]. But only recently the importance of the decomposition in clarifying the non-Abelian dynamics has become appreciated by many authors. Indeed this decomposition has played a crucial role for us to establish the Abelian dominance in Wilson loops in QCD [19], and to clarify the topological structure (in particular the Deligne cohomology) of the non-Abelian gauge theory [17].

An important advantage of the decomposition (1) is that it can actually Abelianize (or more precisely "dualize") the non-Abelian gauge theory [2, 3, 20]. To see this let $(\hat{n}_1,\ \hat{n}_2,\ \hat{n})$ be a right-handed orthonormal basis in $SU(2)$ space and let

$$\vec{X}_\mu = X_\mu^1\,\hat{n}_1 + X_\mu^2\,\hat{n}_2,$$
$$(X_\mu^1 = \hat{n}_1\cdot\vec{X}_\mu,\quad X_\mu^2 = \hat{n}_2\cdot\vec{X}_\mu).$$

With this we have

$$\hat{D}_\mu\vec{X}_\nu = \Big[\partial_\mu X_\nu^1 - g(A_\mu+\tilde{C}_\mu)X_\nu^2\Big]\hat{n}_1$$
$$+\Big[\partial_\mu X_\nu^2 + g(A_\mu+\tilde{C}_\mu)X_\nu^1\Big]\hat{n}_2. \tag{10}$$

So introducing a dual potential B_μ and a complex vector field X_μ by

$$B_\mu = A_\mu + \tilde{C}_\mu,$$
$$X_\mu = \frac{1}{\sqrt{2}}(X_\mu^1 + iX_\mu^2), \tag{11}$$

we can express the Lagrangian explicitly as follows,

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}^2 - \frac{1}{2}|\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu|^2 + igG_{\mu\nu}X_\mu^*X_\nu$$
$$-\frac{1}{2}g^2\Big[(X_\mu^*X_\mu)^2 - (X_\mu^*)^2(X_\nu)^2\Big]$$
$$= -\frac{1}{4}(G_{\mu\nu}+X_{\mu\nu})^2 - \frac{1}{2}|\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu|^2, \tag{12}$$

where

$$G_{\mu\nu} = F_{\mu\nu} + H_{\mu\nu},\quad \hat{D}_\mu X_\nu = (\partial_\mu + igB_\mu)X_\nu,$$
$$X_{\mu\nu} = -ig(X_\mu^*X_\nu - X_\nu^*X_\mu).$$

Clearly this describes an Abelian gauge theory coupled to the charged vector field X_μ. But the important point here is that the Abelian potential B_μ is given by the sum of the electric and magnetic potentials $A_\mu + \tilde{C}_\mu$. In this form the equations of motion of $SU(2)$ QCD is expressed by

$$\partial_\mu(G_{\mu\nu}+X_{\mu\nu}) = igX_\mu^*(\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu)$$
$$-igX_\mu(\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu)^*,$$
$$\hat{D}_\mu(\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu) = igX_\mu(G_{\mu\nu}+X_{\mu\nu}). \tag{13}$$

This shows that one can indeed Abelianize the non-Abelian theory with our decomposition. A remarkable

feature of this Abelian formulation is that here the topological field $\hat{n}$ is replaced by the magnetic potential $\tilde{C}_\mu$ [2, 3].

An important point of this Abelianization is that it is gauge independent, because here we have never fixed the gauge to obtain this Abelian formalism. So one might ask how the non-Abelian gauge symmetry is realized in this Abelian formalism. To discuss this let

$$\begin{aligned}\vec{\alpha} &= \alpha_1\ \hat{n}_1 + \alpha_2\ \hat{n}_2 + \theta\ \hat{n}, \\ \alpha &= \frac{1}{\sqrt{2}}(\alpha_1 + i\ \alpha_2), \\ \vec{C}_\mu &= -\frac{1}{g}\hat{n} \times \partial_\mu \hat{n} = -C^1_\mu \hat{n}_1 - C^2_\mu \hat{n}_2, \\ C_\mu &= \frac{1}{\sqrt{2}}(C^1_\mu + i\ C^2_\mu). \end{aligned} \tag{14}$$

Then the Lagrangian (12) is invariant not only under the active gauge transformation (4) described by

$$\begin{aligned}\delta A_\mu &= \frac{1}{g}\partial_\mu\theta - i(C^*_\mu\alpha - C_\mu\alpha^*), \qquad \delta\tilde{C}_\mu = -\delta A_\mu, \\ \delta X_\mu &= 0, \end{aligned} \tag{15}$$

but also under the following passive gauge transformation described by

$$\begin{aligned}\delta A_\mu &= \frac{1}{g}\partial_\mu\theta - i(X^*_\mu\alpha - X_\mu\alpha^*), \qquad \delta\tilde{C}_\mu = 0, \\ \delta X_\mu &= \frac{1}{g}\hat{D}_\mu\alpha - i\theta X_\mu. \end{aligned} \tag{16}$$

Clearly this passive gauge transformation assures the desired non-Abelian gauge symmetry in the Abelian formalism. This tells that the Abelian theory not only retains the original gauge symmetry, but actually has an enlarged (both the active and passive) gauge symmetries. But we emphasize that this is not the “naive” Abelianization of QCD which one obtains by fixing the gauge. Our Abelianization is a gauge-independent Abelianization.

III. SAVVIDY-NIELSEN-OLESEN EFFECTIVE ACTION: A REVIEW

To calculate the one-loop effective action one must divide the gluon field into two parts, the slow-varying classical background $\vec{B}_\mu$ and the fluctuating quantum part $\vec{Q}_\mu$,

$$\vec{A}_\mu = \vec{B}_\mu + \vec{Q}_\mu, \tag{17}$$

and integrate the quantum part [21, 22]. Of course, the separation of the quantum part from the classical background has to be gauge independent for the effective action to be gauge independent. The decomposition (1) is very useful for this purpose, because it naturally provides the gauge independent separation of the classical background from the quantum fluctuation. Indeed the gauge independent separation follows automatically if we identify the classical background to be the restricted potential $\hat{A}_\mu$ and the quantum fluctuation to be the valence potential $\vec{X}_\mu$.

In the Abelian formalism this means that we can treat B_μ as the classical background and X_μ as the fluctuating quantum part. In this picture the active gauge transformation (15) is viewed as the background gauge transformation and the passive gauge transformation (16) is viewed as the quantum gauge transformation. To calculate the one-loop effective action, we fix the gauge of the quantum gauge transformation by imposing the following gauge condition to X_μ,

$$\begin{aligned}\hat{D}_\mu X_\mu &= 0, \qquad (\hat{D}_\mu X_\mu)^* = 0 \\ \mathcal{L}_{gf} &= -\frac{1}{\xi}|\hat{D}_\mu X_\mu|^2. \end{aligned} \tag{18}$$

Under the gauge transformation (16) the gauge condition depends only on α, so the corresponding Faddeev-Popov determinant is given by

$$M_{FP} = \begin{vmatrix} \dfrac{\delta(\hat{D}_\mu X_\mu)}{\delta\alpha} & \dfrac{\delta(\hat{D}_\mu X_\mu)}{\delta\alpha^*} \\ \dfrac{\delta(\hat{D}_\mu X_\mu)^*}{\delta\alpha} & \dfrac{\delta(\hat{D}_\mu X_\mu)^*}{\delta\alpha^*} \end{vmatrix}. \tag{19}$$

With this gauge fixing the one-loop effective action takes the following form [5, 6, 8, 10],

$$\begin{aligned}\exp\Big[iS_{eff}(B_\mu)\Big] = &\int \mathcal{D}X_\mu \mathcal{D}X^*_\mu \mathcal{D}c_1 \mathcal{D}c^\dagger_1 \mathcal{D}c_2 \mathcal{D}c^\dagger_2 \exp\Big\{ i\int \Big[-\frac{1}{4}(G_{\mu\nu} + X_{\mu\nu})^2 - \frac{1}{2}|\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu|^2 \\ &-\frac{1}{\xi}|\hat{D}_\mu X_\mu|^2 + c^\dagger_1(\hat{D}^2 + g^2 X^*_\mu X_\mu)c_1 - g^2 c^\dagger_1 X_\mu X_\mu c_2 + c^\dagger_2(\hat{D}^2 + g^2 X^*_\mu X_\mu)^* c_2 - g^2 c^\dagger_2 X^*_\mu X^*_\mu c_1 \Big] d^4x \Big\}, \end{aligned} \tag{20}$$

where c_1 and c_2 are the complex ghost fields. To evaluate the integral we notice that the functional determinants of

the valence gluon and the ghost loops are expressed as

$$\mathrm{Det}^{-\frac{1}{2}} K_{\mu\nu} \simeq \mathrm{Det}[-g_{\mu\nu}(\hat{D}\hat{D}) + 2igG_{\mu\nu}], \qquad \mathrm{Det} M_{FP} = \mathrm{Det}[-(\hat{D}\hat{D})]^2. \tag{21}$$

Using the relation

$$G_{\mu\alpha}G_{\nu\beta}G_{\alpha\beta} = \frac{1}{2}G^2 G_{\mu\nu} + \frac{1}{2}(G\tilde{G})\tilde{G}_{\mu\nu}, \qquad (\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma}), \tag{22}$$

we can simplify the functional determinants of the gluon and the ghost loops as follows,

$$\begin{aligned} \ln \mathrm{Det}^{-\frac{1}{2}} K &= \ln \mathrm{Det}[(-\hat{D}^2 + 2a)(-\hat{D}^2 - 2a)] + \ln \mathrm{Det}[(-\hat{D}^2 - 2ib)(-\hat{D}^2 + 2ib)], \\ \ln \mathrm{Det} M_{FP} &= 2 \ln \mathrm{Det}(-\hat{D}^2), \end{aligned} \tag{23}$$

where

$$a = \frac{g}{2}\sqrt{\sqrt{G^4 + (G\tilde{G})^2} + G^2}, \qquad b = \frac{g}{2}\sqrt{\sqrt{G^4 + (G\tilde{G})^2} - G^2}.$$

Notice that two determinants $\mathrm{Det}(-\hat{D}^2 \pm 2a)$ (and $\mathrm{Det}(-\hat{D}^2 \pm 2ib)$) correspond to two spin orientations of the valence gluon.

Savvidy has chosen a covariantly constant color magnetic field as the classical background [4, 5, 6]

$$\begin{aligned} \vec{B}_\mu &= \frac{1}{2}H_{\mu\nu}x_\nu \hat{n}_0, \qquad \vec{G}_{\mu\nu} = H_{\mu\nu}\hat{n}_0, \\ \bar{D}_\mu \vec{G}_{\mu\nu} &= 0, \end{aligned} \tag{24}$$

where $H_{\mu\nu}$ is a constant magnetic field and $\hat{n}_0$ is a constant unit isovector. With this one has

$$\begin{aligned} \Delta S &= i \ln \mathrm{Det}[(-\hat{D}^2 + 2gH)(-\hat{D}^2 - 2gH)] \\ &\quad -2i \ln \mathrm{Det}(-\hat{D}^2), \\ H &= \sqrt{\frac{H_{\mu\nu}^2}{2}}. \end{aligned} \tag{25}$$

One can evaluate the functional determinants using Schwinger's proper time method [23], and find

$$\begin{aligned} \Delta\mathcal{L} = \frac{1}{16\pi^2}\int_0^\infty \frac{dt}{t^2}\frac{gH/\mu^2}{\sinh(gHt/\mu^2)}\Big[\exp(-2gHt/\mu^2) \\ + \exp(+2gHt/\mu^2)\Big], \end{aligned} \tag{26}$$

where μ is a dimensional parameter. The integral has a severe infra-red divergence, and to perform the integral we have to regularize the infra-red divergence first. Let us regularize it with the ζ-function regularization. From the definition of the generalized ζ-function [24]

$$\begin{aligned} \zeta(s,\lambda) &= \sum_{n=0}^{\infty}\frac{1}{(n+\lambda)^s} \\ &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}\exp(-\lambda x)}{1-\exp(-x)}dx, \end{aligned} \tag{27}$$

we have

$$\begin{aligned} \Delta\mathcal{L} &= \lim_{\epsilon\to 0}\frac{1}{16\pi^2}\int_0^\infty \frac{dt}{t^{2-\epsilon}}\frac{gH\mu^2}{\sinh(gHt/\mu^2)}\Big[\exp(-2gHt/\mu^2) + \exp(+2gHt/\mu^2)\Big] \\ &= \lim_{\epsilon\to 0}\frac{gH\mu^2}{8\pi^2}\int_0^\infty \frac{dt}{t^{2-\epsilon}}\frac{\exp(-3gHt/\mu^2) + \exp(+gHt/\mu^2)}{1-\exp(-2gHt/\mu^2)} \\ &= \lim_{\epsilon\to 0}\frac{g^2H^2}{4\pi^2}(\frac{2gH}{\mu^2})^{-\epsilon}\Gamma(\epsilon-1)\Big[\zeta(\epsilon-1,\frac{3}{2}) + \zeta(\epsilon-1,-\frac{1}{2})\Big] \\ &= \lim_{\epsilon\to 0}\frac{g^2H^2}{4\pi^2}(1-\epsilon\ln\frac{2gH}{\mu^2})\big(\frac{1}{\epsilon}-\gamma+1\big)\Big[\big(\zeta(-1,\frac{3}{2}) + \zeta(-1,-\frac{1}{2})\big) + \epsilon\big(\zeta'(-1,\frac{3}{2}) + \zeta'(-1,-\frac{1}{2})\big)\Big] \\ &= \frac{11g^2H^2}{48\pi^2}\big(\frac{1}{\epsilon}-\gamma+1-\ln\frac{2gH}{\mu^2}\big) - \frac{g^2H^2}{4\pi^2}\big(2\zeta'(-1,\frac{3}{2}) - i\frac{\pi}{2}\big), \end{aligned} \tag{28}$$

where $\zeta' = \dfrac{d\zeta}{ds}(s,\lambda)$, and we have used the fact [24]

$$\zeta(-1,\frac{3}{2}) = \zeta(-1,-\frac{1}{2}) = -\frac{11}{24}, \qquad \zeta'(-1,-\frac{1}{2}) = \zeta'(-1,\frac{3}{2}) - i\frac{\pi}{2}.$$

So, with the ultra-violet regularization by modified minimal subtraction we obtain the SNO effective action

$$\mathcal{L}_{eff} = -\frac{H^2}{2} - \frac{11g^2}{48\pi^2}H^2(\ln\frac{gH}{\mu^2} - c) + i\frac{g^2}{8\pi}H^2,$$
$$c = 1 - \ln 2 - \frac{24}{11}\zeta'(-1, \frac{3}{2}) = 0.94556.... \quad (29)$$

The real part of the effective action has a non-trivial SNO vacuum at $< H > \neq 0$. Unfortunately the effective action contains the well-known imaginary part which destabilzes the SNO vacuum.

IV. GAUGE INVARIANT CALCULATION OF EFFECTIVE ACTION

Notice that the imaginary part of the effective action originates from the determinant $\mathrm{Det}(-\hat{D}^2 - 2gH)$, which corresponds to the gluon loop whose spin is anti-parallel to the magnetic field [6]. Indeed in the Abelian formalism the calculation of the functional determinants $\mathrm{Det}(-\hat{D}^2 \pm 2gH)$ in (25) amounts to the calculation of the energy eigenvalues of a massless charged vector field (the valence gluon) in a constant external magnetic field $H_{\mu\nu}$. Choosing the direction of the magnetic field to be the z-direction, one obtains the well-known eigenvalues

$$E = 2gH(n + \frac{1}{2}) + k^2 \pm 2gH,$$
$$H = H_{12}, \quad (30)$$

where k is the momentum of the eigen-function in z-direction. Notice that the $\pm$ signature correspond to the spin $S_3 = \pm 1$ of the valence gluon. So, when $n = 0$, the eigen-functions with $S_3 = -1$ have an imaginary energy when $k^2 < gH$, and thus become tachyons which violate the causality. And these tachyonic eigenstates create the instability of the magnetic background and the imaginary part in the effective action [6, 8, 10].

The existence of the tachyonic modes in the functional determinant tells that the calculation of the effective action violates the causality. This means that we must exclude the unphysical tachyonic modes from the functional determinant. The question is how. A natural answer is to impose the causality in the calculation of the effective action [8, 10]. To show this we go to the Minkowski time with the Wick rotation, and find that (26) changes to

$$\Delta\mathcal{L} = -\frac{1}{16\pi^2}\int_0^\infty \frac{dt}{t^2}\frac{gH/\mu^2}{\sin(gHt/\mu^2)}\Big[\exp(-i2gHt/\mu^2) + \exp(+i2gHt/\mu^2)\Big], \quad (31)$$

In this form the infra-red divergence has disappeared, but now we face an ambiguity in choosing the correct contours of the integrals in (31). But this ambiguity can be resolved by causality. Indeed the standard causality argument requires us to identify $2gH$ in the first integral as $2gH - i\epsilon$, but in the second integral as $2gH + i\epsilon$. This tells that the poles in the first integral in (31) should lie above the real axis, but the poles in the second integral should lie below the real axis [8, 10]. With this causality requirement the two integrals become complex conjugate to each other. This tells that the effective action must be explicitly real, without any imaginary part.

The fact that the causality removes the tachyonic modes in the functional determinant is perhaps not surprising. What is surprizing is that a completely independent principle, the gauge invariance, can also remove the tachyonic modes [11]. To see this, notice that the Savvidy background (24) is not gauge invariant. Indeed $\vec{G}_{\mu\nu}$ must be gauge covariant. So one can change $\vec{G}_{\mu\nu}$ to $-\vec{G}_{\mu\nu}$, and thus $H_{\mu\nu}$ to $-H_{\mu\nu}$, by a gauge transformation (with the color reflection of $\hat{n}_0$ to $-\hat{n}_0$). In this case $\mathrm{Det}(-\hat{D}^2 + 2gH)$ changes to $\mathrm{Det}(-\hat{D}^2 - 2gH)$, and vise versa, under the gauge transformation. On the other hand the gauge transformation does not affect the gluon spin. This means that one can change the direction of magnetic field with respect to the spin polarization direction of gluon by a gauge transformation. In other words the spin polarization direction of the gluon with respect to the magnetic background is a gauge artifact. More importantly the eigenvalues of the $S_3 = +1$ gluon shift nagatively, and those of the $S_3 = -1$ gluon shift positively, by a factor $2gH$ under the gauge transformation. And obviously only the eigenvalues which are invariant under this transformation should qualify to be gauge invariant. This means that the gauge invariant eigenstates are those which are independent of the spin orientation of the valence gluon which appear in both $S_3 = +1$ and $S_3 = -1$ simultaneously.

This is shown schematically in Fig. 1, where (A) transforms to (B) under the color reflection. This clearly tells that the tachyonic modes which caused the instability of the SNO vacuum is not gauge invariant, and thus should not be included in the calculation of the gauge-invariant functional determinants. This means that a gauge invariant calculation of the effective action must produce a stable magnetic condensation.

One might think (incorrectly) that the eigenvalues of the functional determinants do not change because $\mathrm{Det}(-\hat{D}^2 \pm 2gH)$ remain unchanged under the gauge transformation, even though $\mathrm{Det}(-\hat{D}^2 + 2gH)$ changes

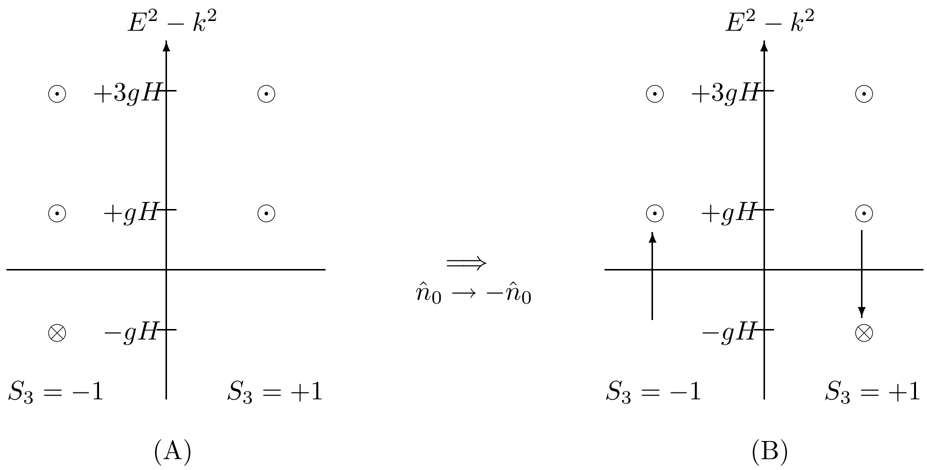

FIG. 1: The eigenvalues of the functional determinant of the gluon loop. When the gluon spin is anti-parallel to the magnetic field ($S_3 = -1$), the ground state (with $n = 0$) becomes tachyonic when $k^2 < gH$. Notice, however, that under the color reflection of $\hat{n}_0$ to $-\hat{n}_0$ H changes to $-H$ so that the eigenvalues change from (A) to (B). This shows that the spin polarization direction of gluon with respect to the magnetic field is a gauge artifact. This excludes the tachyons from the functional determinant.

to $\mathrm{Det}(-\hat{D}^2 - 2gH)$ and vise versa. This is wrong, because here one must calculate the eigenvalues of the spin-up gluon and spin-down gluon separately. This is very important. And the determinant for the spin-up gluon $\mathrm{Det}(-\hat{D}^2 + 2gH)$ changes to $\mathrm{Det}(-\hat{D}^2 - 2gH)$ and the determinant for the spin-down gluon $\mathrm{Det}(-\hat{D}^2 - 2gH)$ changes to $\mathrm{Det}(-\hat{D}^2 + 2gH)$. So the eigenvalues change, and for each spin polarization only the positive eigenvalues remain invariant under the gauge transformation.

With the gauge invariant calculation of the the functional determinants, the effective action (26) changes to

$$\Delta\mathcal{L} = \frac{1}{16\pi^2} \int_0^\infty \frac{dt}{t^2} \frac{gH/\mu^2}{\sinh(gHt/\mu^2)} \Big[\exp(-2gHt/\mu^2) + \exp(-2gHt/\mu^2) \Big], \quad (32)$$

which has no infra-red divergence at all. This precludes the necessity to make any infra-red regularization. ¿From this we have [8, 10, 11]

$$\mathcal{L}_{eff} = -\frac{H^2}{2} - \frac{11g^2}{96\pi^2} H^2 (\ln \frac{gH}{\mu^2} - c), \quad (33)$$

which clearly has no imaginary part.

A physical way to understand the above result is to remember that the one-loop effective action is nothing but the vacuum to vacuum amplitude in the presence of the classical background,

$$\exp\Big[iS_{eff}(\vec{B}_\mu) \Big] = < \Omega_+ | \ \Omega_- > \Big|_{\vec{B}_\mu} = \sum_{|n_i>} < \Omega_+ | \ n_i >< n_i \ | \ \Omega_- > \Big|_{\vec{B}_\mu}, \quad (34)$$

where $|\Omega >$ is the vacuum and $|n_i >$ is a complete set of orthonormal states of QCD. In this picture the gluon loop integral corresponds to the summation of the complete set of states. And obviously the complete set should not include the tachyons, unless one wants to assert that the physical spectrum of QCD must contain the unphysical tachyons which violate (not only the causality but also) the gauge invariance. This justifies the exclusion of the tachyons in the calculation of the functional determinant.

The above discussion tells that $SU(2)$ QCD is able to generate a stable magnetic condensation. But the stability of the Savvidy vacuum has been so controversal that one might like to see more evidence to support the stable magnetic condensation. A best way to do this is to construct a classical magnetic background which has no unstable modes. In the following we will show that an axially symmetric monopole string and anti-monopole string pair has no unstable modes under the quantum fluctuation, if the distance between two strings is small enough, smaller than a critical value.

V. INSTABILITY OF MONOPOLE-ANTIMONOPOLE BACKGROUND

Before we discuss the stability of monopole-antimonopole pair, we first review the instability of the Wu-Yang monopole because two problems are closely related [14]. In our notation the Wu-Yang monopole of charge q/g is described by [13]

$$\vec{A}_\mu = -\frac{1}{g} \hat{n} \times \partial_\mu \hat{n}, \qquad \hat{n} = \begin{pmatrix} \sin\theta\cos q\phi \\ \sin\theta\sin q\phi \\ \cos\theta \end{pmatrix}, \quad (35)$$

where (r, θ, ϕ) is the spherical coordinates and q is an integer. But in the Abelian formalism it is more convenient for us to describe the monopole in terms of the magnetic potential $\tilde{C}_\mu$,

$$\tilde{C}_\mu = \frac{q}{g}(\cos\theta - 1)\partial_\mu \phi, \tag{36}$$

which is nothing but the Dirac's monopole potential.

To study the stability of the monopole background we consider the functional determinant of the monopole which provides the one-loop correction to the effective action

$$\mathrm{Det}\, K = \mathrm{Det}(-\tilde{D}^2 \pm 2\frac{q}{r^2}). \tag{37}$$

Notice that the functional determinant is precisely the one we introduced in (25), exept that here H is given by the magnetic field strength of the monopole.

Now, the absence or presence of negative modes of the operator K implies stability or instability of the classical background against small fluctuations of the gauge potential. To calculate the eigenvalues of the operator K one can rewrite the eigenvalue equation as the following Schroedinger type equation acting on a complex scalar field Ψ,

$$\begin{aligned}
&K\Psi(r,\theta,\phi) = E\Psi(r,\theta,\phi), \\
&K = -\Delta - 2\frac{iq}{r^2\sin^2\theta}\cos\theta\partial_\phi + \frac{q^2}{r^2}\cot^2\theta \pm 2\frac{q}{r^2}, \\
&\Delta = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 \\
&\quad = \frac{1}{r^2}\partial_r(r^2\partial_r) - \frac{\hat{L}^2}{r^2}, \\
&\hat{L}^2 = -\Big(\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2\Big),
\end{aligned} \tag{38}$$

where $\hat{L}$ is the angular momentum operator. Notice that here again the $\pm$ signatures represent two spin orientations of the valence gluon.

With

$$\Psi(r,\theta,\phi) = R(r)Y(\theta,\phi), \tag{39}$$

one obtains the equation for the angular eigenfunction $Y(\theta,\phi)$ from (38)

$$\Big(\hat{L}^2 - \frac{2iq\cos\theta}{\sin^2\theta}\partial_\phi + q^2\cot^2\theta\Big)Y(\theta,\phi) = \lambda Y(\theta,\phi). \tag{40}$$

Moreover, with

$$Y(\theta,\phi) = \sum_{m=-\infty}^{+\infty}\Theta_m(\theta)\Phi_m(\phi), \qquad \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}\exp(im\phi). \tag{41}$$

one can reduce (40) to

$$\Big(-\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{(m+q\cos\theta)^2}{\sin^2\theta}\Big)\Theta = \lambda\Theta. \tag{42}$$

This is exactly the eigenvalue equation for the monopole harmonics which has been well-studied in the literature [12]. ¿From the equation one obtains the following expression for the monopole harmonics and the corresponding eigenvalue spectrum

$$\begin{aligned}
&Y_{qjm}(\theta,\phi) = \Theta_{qjm}(\theta)\Phi_m(\phi), \\
&\Theta_{qjm}(\theta) = (1-\cos\theta)^{\gamma_+}(1+\cos\theta)^{\gamma_-} \times P_k(\cos\theta), \\
&\lambda = (k+p)(k+p+1) - q^2 = j(j+1) - q^2, \\
&p = \gamma_+ + \gamma_-, \quad \gamma_+ = \frac{|m+q|}{2}, \quad \gamma_- = \frac{|m-q|}{2}, \\
&j = k + p = k + \frac{1}{2}(|m-q| + |m+q|), \\
&k = 0,1,2,...,
\end{aligned} \tag{43}$$

where $P_k(x)$ is the Legendre polynomial of order k. The quantum number j is analogous to the orbital angular momentum quantum number l of the standard spherical harmonics Y_{lm}, except that here j starts from a non-zero integer value for a non-vanishing monopole charge q.

Now, consider the equation for the radial eigenfunction

$$\Big(\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr}) - \frac{1}{r^2}\big[j(j+1) - q^2\big] \mp 2\frac{q}{r^2} + E\Big)R(r) = 0. \tag{44}$$

With $R(r) = \frac{1}{r}\chi(r)$ one obtains

$$\Big(\frac{d^2}{dr^2} - \frac{1}{r^2}\big[j(j+1) - q^2 \pm 2q\big] + E\Big)\chi(r) = 0. \tag{45}$$

The equation has a general solution in terms of Bessel functions of the first kind $J_\nu(z)$

$$\chi(r) = \sqrt{r}\Big[C_1 J_{-\nu}(\sqrt{E}r) + C_2 J_\nu(\sqrt{E}r)\Big], \qquad \nu = \frac{1}{2}\sqrt{1 + 4[j(j+1) - q^2 \pm 2q]}, \tag{46}$$

where C_i $(i = 1,2)$ are the integration constants. For positive values of ν and E the finite solutions oscillating at the infinity and vanishing at the origin are given by $C_1 = 0$. The negative eigenvalues of E can come only from (45) with the lower negative sign (which corresponds to the operator $-\tilde{D}^2 - 2q/r^2$) and the lowest value of $j = 1$ with $q = 1$. In this case we have to solve the equation

$$\Big(\frac{d^2}{dr^2} + \frac{1}{r^2} + E\Big)\chi = 0, \tag{47}$$

which is nothing but the one-dimensional Schroedinger equation for the attractive potential $-1/r^2$. So the the monopole stability problem is reduced to the well known one-dimensional quantum mechanical problem with an attractive potential proportional to $-1/r^2$ [25].

With $\chi(r) = \sqrt{r}y(r)$ and $z = \sqrt{E}r$, (47) reduces to

$$\Big(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + (z^2 + \frac{3}{4})\Big)y = 0. \tag{48}$$

In this form the energy dependence of the equation disappears completely, so that we have no condition of discreteness for the eigenvalue spectrum. The analytic solution to the equation is given by the Bessel function of the first kind

$$y = J_{i\sqrt{3/4}}(z). \tag{49}$$

For the real argument z (i.e. for the positive energies E), the real and imaginary parts of the solution provide two independent solutions vanishing at the infinity and oscillating at the origin. For negative energies (i.e. for the pure imaginary z), the real part of the Bessel function diverges at the infinity, so that only the imaginary part of the Bessel function becomes a physical solution. This solution has a continuum of negative eigenvalue spectrum, and is oscillating near the origin and approaches to zero exponentially at the infinity. Both solutions, for positive and negative energies, the radial eigenfunction $R(r)$ behaves like

$$R(r) \simeq \frac{\sin\log(\sqrt{|E|}r) + \text{const}}{\sqrt{r}}, \tag{50}$$

near the origin. The solutions have infinite number of zeros approaching the point $r = 0$, so that for the negative energies the valence gluon moving around the monopole must fall down to the center [14]. For higher monopole charges the qualitative picture remains the same, and one still has a continious bound state energy spectrum.

The above analysis implies that the undesired attractive potential proportional to $-1/r^2$ in (45) vanishes when $q = 0$, in which case j starts from zero. This can serve as a hint that one might expect the absence of negative modes only for a magnetic background with vanishing monopole charge. So it is important to check the stability of magnetic background with vanishing monopole charge.

A simplest magnetic background which has a vanishing monopole charge is a monopole-antimonopole pair. But in the following we show that the monopole-antimonopole background is not stable. Consider a monopole-antimonopole background where the monopole and anti-monopole are located at the point ($x = y = 0, z = a$) and at the origin ($x = y = z = 0$). The magnetic fields of the monopole and anti-monopole are expressed by

$$\vec{H}_+ = \frac{\vec{r} - \vec{a}}{|\vec{r} - \vec{a}|^3}, \qquad \vec{H}_- = -\frac{\vec{r}}{r^3},$$

$$\vec{H} = \vec{H}_+ + \vec{H}_-,$$
$$H = \frac{1}{r^2}\Big(1 + \frac{r^4}{r'^4} - 2\frac{r^3}{r'^3}(1 - \frac{a}{r}\cos\theta)\Big)^{\frac{1}{2}}$$
$$r' = r(1 - 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2})^{\frac{1}{2}}. \tag{51}$$

The corresponding magnetic potential $\tilde{C}_\mu$ is given by

$$\tilde{C}_\mu = \Big[(1 - \frac{r}{r'})\cos\theta + \frac{a}{r'}\Big]\partial_\mu\phi. \tag{52}$$

With the magnetic background we can repeat the above stability analysis to see whether the monopole-antimonopole pair is stable or not.

The above background gives the following eigenvalue equation similar to (38)

$$\begin{aligned}\Big(&\frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 \\ &+\frac{2i}{r^2\sin^2\theta}\big[(1 - \frac{r}{r'})\cos\theta + \frac{a}{r'}\big]\partial_\phi \\ &-\frac{1}{r^2\sin^2\theta}\big[(1 - \frac{r}{r'})\cos\theta + \frac{a}{r'}\big]^2 \mp 2H + E\Big)\Psi \\ &= 0.\end{aligned} \tag{53}$$

Here again the $\mp H$ term correspond to two spin orientations of the valence gluon with respect to the magnetic field. Due to the axial symmetry we can put

$$\Psi(r,\theta,\phi) = \sum_{m=-\infty}^{+\infty} F_m(r,\theta)\Phi_m(\phi), \tag{54}$$

and obtain the equation for $F_m(r,\theta)$

$$\begin{aligned}&\Big(\frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) - U + E\Big)F = 0, \\ &U = \frac{1}{r^2\sin^2\theta}\big[(1 - \frac{r}{r'})\cos\theta + \frac{a}{r'} + m\big]^2 \pm 2H.\end{aligned} \tag{55}$$

To find the energy spectrum let us consider the equation near the origin. Keeping the leading terms in the limit r goes to zero we can approximate the equation as

$$\begin{aligned}\Big(&\frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) \\ &-\frac{(m + 1 + \cos\theta)^2}{r^2\sin^2\theta} \mp \frac{2}{r^2} + E\Big)F \simeq 0.\end{aligned} \tag{56}$$

The equation is separable, and the angular part of the eigenfunction is expressed by the monopole harmonic function Θ_{1jm} which satisfies the eigenvalue equation

$$\begin{aligned}\Big(&\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) - \frac{(m + 1 + \cos\theta)^2}{\sin^2\theta} \\ &+\big[j(j+1) - 1\big]\Big)\Theta_{1jm} = 0,\end{aligned} \tag{57}$$

where j is given by

$$j = \frac{|m|}{2} + \frac{|m+2|}{2}. \tag{58}$$

Clearly the lowest value of j is 1, which tells that the radial part of the potential U still has an attractive potential proportional to $-1/r^2$. ¿From this we conclude that the monopole-antimonopole pair has to be unstable.

The lesson from this analysis is clear. Although the total magnetic charge of monopole-antimonopole pair is zero, the potential of the pair near the origin still retains the undesirable attractive potential which makes the magnetic background unstable.

VI. AXIALLY SYMMETRIC MONOPOLE STRING BACKGROUND

Now we consider the axially symmetric monopole string, which can be regarded as an infinite string carrying homogeneous monopole charge density along the string. The magnetic field strength of the axially symmetric monopole string can be written in a simple form in the cylindrical coordinates (ρ, ϕ, z)

$$\vec{H} = \frac{\alpha}{\rho}\hat{\rho},$$
$$\tilde{C}_\mu = -\alpha(z+\tau)\partial_\mu\phi, \tag{59}$$

where α is the monopole charge density and τ is an arbitrary constant which represents the translational invariance of the magnetic field along the z-axis. Just like the monopole solution (35) the above monopole string forms a classical solution of $SU(2)$ QCD, because it satisfies the equation of motion (13). But in the following we will assume $\alpha = 1$ and $\tau = 0$ for simplicity, since this will not affect the stability analysis. Notice that here α has the dimension of a mass, so that setting $\alpha = 1$ amounts to fixing the scale in the unit of $1/\alpha$

Again we consider the eigenvalue problem for the operator K

$$K\Psi(\rho,\phi,z) = E\Psi(\rho,\phi,z). \tag{60}$$

With

$$\Psi = \sum_{m=-\infty}^{+\infty} F_m(\rho,z)\Phi_m(\phi), \tag{61}$$

and repeating the steps of the previous section we obtain the following eigenvalue equation

$$F_{\rho\rho} + \frac{1}{\rho}F_\rho + F_{zz} - \Big[\frac{(m-z)^2}{\rho^2} \pm \frac{2}{\rho} - E\Big]F = 0. \tag{62}$$

One can put $m = 0$ (shifting z to $z+m$) and simplify the equation

$$F_{\rho\rho} + \frac{1}{\rho}F_\rho + F_{zz} - \Big[\frac{z^2}{\rho^2} \pm \frac{2}{\rho} - E\Big]F = 0. \tag{63}$$

The quantum mechanical potential of this equation behaves like $\pm 2/\rho$ near $\rho = 0$. So we still have an undesired attractive potential $-2/\rho$. This implies two things. First, the attractive interaction of the axially symmetric monopole string background is less severe than the attractive interaction of the spherically symmetric monopole background. So we can expect the absence of continuous negative energy spectrum for the axially symmetric monopole string background. Secondly, the attractive potential $-2/\rho$ tells that the monopole string background must still be unstable, because it indicates the existence of discrete bound states with negative energy.

To confirm this we make a qualitative estimate of the negative energy eigenvalues of (63). We look for a solution which has the form

$$F(\rho,z) = \sum_{n=0}^{\infty} f_n(\rho)Z_n(x),$$
$$Z_n(x) = \exp(-\frac{x^2}{2})H_n(x),$$
$$x = \frac{z}{\sqrt{\rho}}, \tag{64}$$

where $H_n(x)$ is the Hermite polynomial. Notice that $Z_n(x)$ forms a complete set of eigenfunctions of the harmonic oscillator,

$$\Big(\frac{d^2}{dx^2} - x^2\Big)Z_n(x) = -(2n+1)Z_n(x). \tag{65}$$

Substituting (64) into (63) we obtain

$$\sum_{n=0}^{\infty}\Big\{\Big(\frac{d^2 f_n}{d\rho^2} + (\frac{1}{\rho} + \frac{z^2}{\rho^2})\frac{df_n}{d\rho}\Big)H_n - \frac{z}{\rho\sqrt{\rho}}\frac{df_n}{d\rho}\frac{dH_n}{dx} + f_n\Big(\frac{z^2}{4\rho^3}\frac{d^2H_n}{dx^2} - \frac{z}{4\rho^2\sqrt{\rho}}(1-\frac{z^2}{\rho^2})\frac{dH_n}{dx}\Big) + \Big(\frac{z^4}{4\rho^4} - \frac{z^3}{4\rho^3\sqrt{\rho}} + \frac{z^2}{\rho^3} - \frac{2n+1\pm 2}{\rho} + E\Big)f_nH_n\Big\} = 0. \tag{66}$$

Using the reccurence relations and orthogonality properties of Hermite polynomials one can derive the equations for $f_n(\rho)$ from (66)

$$\Big(\frac{d^2}{d\rho^2} + \frac{2n+3}{2\rho}\frac{d}{d\rho} + \frac{4n^2-2n-1}{16\rho^2} - \frac{2n+1\pm 2}{\rho} + E\Big)f_n$$
$$= -\frac{1}{64\rho^2}f_{n-4} + \frac{1}{32\rho^2}f_{n-3} - \frac{1}{4\rho}\Big(\frac{d}{d\rho} + \frac{n-1}{4\rho}\Big)f_{n-2} + \frac{3}{16\rho^2}f_{n-1} + \frac{3(n+1)^2}{8\rho^2}f_{n+1}$$

$$+\frac{(n+1)(n+2)}{\rho}\Big(\frac{d}{d\rho}-\frac{3n+2}{4\rho}\Big)f_{n+2}$$
$$+\frac{(n+1)(n+2)(n+3)}{4\rho^2}\Big(f_{n+3}$$
$$-3(n+4)f_{n+4}\Big), \quad (67)$$

where $f_n = 0$ for negative integers n. So we have infinite number of equations for infinite number of unknown functions $f_n(\rho)$.

Notice that the left hand side of the last equation is a second order differential equation for f_n with the quantum mechanical potential

$$U = \frac{2n+1\pm 2}{\rho}. \quad (68)$$

The potential becomes attractive only if $n = 0$, so that in a first approximation we can expect that the negative energy eigenvalues will originate mainly from the lowest bound state with $n = 0$ of the harmonic oscillator part. So we can hope that by neglecting all f_n with $n \neq 0$ we can still get an approximate qualitative solution for f_0. In this approximation the equation reduces to the following simple equation

$$\Big(\frac{d^2}{d\rho^2}+\frac{3}{2\rho}\frac{d}{d\rho}+\frac{1}{\rho}-\frac{1}{16\rho^2}+E\Big)f_0 = 0. \quad (69)$$

The solution to this equation has a new integer quantum number k,

$$f_{0,k}(\rho) = \rho^s \exp\big(-\sqrt{|E_k|}\rho\big)\sum_{l=0}^{l=k} a_l\rho^l,$$
$$s = \frac{\sqrt{2}-1}{4},$$
$$E_k = -\frac{1}{(2k+2s+3/2)^2},$$
$$a_{l+1} = \frac{\sqrt{|E_k|}(2l+2s+3/2)-1}{(l+1)(l+2s+3/2)}a_l, \quad (70)$$

With this we may express the corresponding eigenfunction Ψ_k as

$$\Psi_k(\rho,\phi,z) = N_k \exp\big(-\frac{z^2}{2\rho}\big)f_{0,k}(\rho), \quad (71)$$

where N_k is a normalization constant. From this we find the lowest energy eigenvalues as follows

$$E_0 = -0.343...,$$
$$E_1 = -0.073...,$$
$$E_2 = -0.031...,$$
$$E_3 = -0.017...,$$
$$E_4 = -0.011.... \quad (72)$$

This confirms that the axially symmetric monopole string background is indeed unstable.

Surprisingly, we find that the approximate solution (71) can also be obtained as an exact solution of variational method with the trial function $\tilde{F}$ of the form

$$\tilde{F}(\rho,z) = N\rho^s \exp\big(-\beta_k\rho-\gamma\frac{z^2}{2\rho}\big)$$
$$\times\sum_{l=0}^{l=k} a_l\rho^l, \quad (73)$$

where s, β_k, γ, a_l are treated as variational parameters. In other words, the variational minimum of the energy functional with the above trial function is provided exactly by the solution (71) with (72).

The knowledge of the solution (71) allows us to develope the perturbation theory to find a more accurate solution. To do this we split the original equation (62) into two parts

$$(H_0+H_1)F(\rho,x) = -EF(\rho,x),$$
$$H_0 = \partial_{\rho\rho}+\frac{3}{2\rho}\partial_\rho+\frac{1}{\rho}-\frac{1}{16\rho^2}+\frac{1}{\rho}\partial_{xx}+\frac{1-x^2}{\rho},$$
$$H_1 = \frac{x^2}{4\rho^2}\partial_{xx}+\frac{x}{4\rho^2}\partial_x-\frac{x}{\rho}\partial_{\rho x}-\frac{1}{2\rho}+\frac{1}{16\rho^2}, \quad (74)$$

and treat H_1 as a perturbation. Looking for eigenfunctions of the non-perturbed Hamiltonian H_0 in the form of

$$F(\rho,x) = f(\rho)Z(x), \quad (75)$$

we can separate the variables ρ and x

$$\Big(\partial_{\rho\rho}+\frac{3}{2\rho}\partial_\rho-\frac{1}{16\rho^2}+\frac{1-2n}{\rho}\Big)f(\rho) = -Ef(\rho),$$
$$(\partial_{xx}-(1-x^2))Z_n(x) = -2nZ_n(x),$$
$$Z_n(x) = \exp\big(-\frac{x^2}{2}\big)H_n(x),$$
$$(n = 0,1,2,...). \quad (76)$$

The equation for $f(\rho)$ has solutions with a discrete negative spectrum when $n = 0$ and continious positive spectrum when n is a positive integer. For $n = 0$ the corresponding eigenfunctions $f_{0,k}(\rho)$ and discrete energy spectrum E_k are given by (70) and(72). For positive integer n the physical solution is numerated by n and continious positive parameter E, and expressed in terms of the confluent hypergeometric function $\mathcal{F}(a,b,\zeta)$ of Kummer

$$f_{n,E}(\rho) = \exp\big(-i\sqrt{E}\rho\big)\rho^{\frac{\sqrt{2}-1}{4}}$$
$$\times\mathcal{F}(\frac{\sqrt{2}+2}{4}+i\frac{2n+1}{2\sqrt{E}},\frac{\sqrt{2}+1}{2},2i\sqrt{E}\rho). \quad (77)$$

Notice that this solution is real and has a correct finite limit at $E = 0$ and correct asymptotic behaviour at $\rho = \infty$

$$f_{n,E} \to \rho^{-\frac{3}{4}}. \quad (78)$$

The eigenfunctions $f_{n,E}$ can be normalized with the standard normalization prescription

$$\int_0^\infty f^*_{n,E'} f_{n,E} \rho^{\frac{3}{2}} d\rho = 2\pi\delta(E' - E). \tag{79}$$

The solution $f_{n,E}$ is numbered by two quantum numbers n and E, where E takes discrete values $E = E_k$ for negative energies and continious values for positive ones, and forms a complete set of eigenfunctions.

In the lowest order of perturbation theory we have, for the bound states (i.e., for $n = 0$), the same energy spectrum E_k as given by (70). The higher order corrections to the energy eigenvalues $E_k = E_{n=0,E=E_k}$ of the bound states are given by the perturbative calculation

$$\begin{aligned} E^{(1)}_{0,E_k} &=< 0, E|H_1|0, E > \\ = & \int \rho^{\frac{3}{2}} f^*_{0,E}(\rho, x) f_{0,E}(\rho, x) d\rho dx d\phi, \\ E^{(2)}_{0,E_k} &= \int \frac{dE'}{E_k - E'} \\ \times \sum_n & < 0, E_k|H_1^\dagger|n', E' >< n', E'|H_1|0, E_k > . \end{aligned} \tag{80}$$

Notice that all first order corrections $E^{(1)}_{0,E_k}$ are vanishing identically due to the fact that the solution (71) turns out to be an exact solution of the variational method. Solving numerically the equation (74) we obtain

$$\begin{aligned} E_0 &= -0.545..., \\ E_1 &= -0.093..., \\ E_2 &= -0.036..., \\ E_3 &= -0.019..., \\ E_4 &= -0.011.... \end{aligned} \tag{81}$$

As we can see the higher order corrections change the ground state energy most significantly, but the qualitative features of the solution remain the same. From this we conclude that the approximate solution (71) provides a good qualitative estimation of the energy spectrum for us to analyse the vacuum stability of the axially symmetric monopole string background.

The lesson that we learn from this analysis is that the monopole string background is still unstable, but it does improve the instability of the spherically symmetric monopole background significantly. This is because here the unstable attractive force becomes milder. This raises a hope that a pair of the axially symmetric monopole and anti-monopole strings might form a stable magnetic background. In the following we prove that this is indeed the case.

VII. A STABLE MAGNETIC BACKGROUND: AXIALLY SYMMETRIC MONOPOLE STRING AND ANTI-MONOPOLE STRING PAIR

The main idea how to construct a stable magnetic background is straightforward now. Consider a pair of axially symmetric monopole and anti-monopole strings which are orthogonal to the xy-plane and separated by a distance a. Now, due to the opposite directions of the magnetic fields of the monopole and anti-monopole strings the total quantum mechanical potential $U(\rho)$ in the eigenvalue equation falls down when $\rho \to \infty$ as $U(\rho) \to O(1/\rho^2)$. By decreasing the distance a we can decrease the effective size of the quantum mechanical potential well (as we will show below), so that the bound state energy levels will be pushed out from the well at some finite critical value of a. This implies that the bound states will have disappeared completely at small enough a.

To show this, consider an axially symmetric monopole and anti-monopole string pair located at ($\rho = a/2, \phi = 0$) and ($\rho = a/2, \phi = \pi$) in cylindrical coordinates. The magnetic field strengths $\vec{H}_\pm$ for the monopole and anti-monopole strings are given by

$$\begin{aligned} \vec{H}_\pm &= \pm \frac{\alpha}{\rho_\pm} \hat{\rho}_\pm, \\ \vec{\rho}_\pm &= \vec{\rho} \pm \frac{\vec{a}}{2}, \\ \rho_\pm^2 &= \rho^2 \pm a\rho\cos\phi + \frac{a^2}{4}, \end{aligned} \tag{82}$$

where $\vec{a}$ is the two-dimensional vector starting from the anti-monopole string to the monopole string in xy-plane. Again from now on we will assume $\alpha = 1$ without loss of generality. The total magnetic field is given by

$$\begin{aligned} \vec{H} &= \vec{H}_+ + \vec{H}_-, \\ H_\rho &= \frac{\rho - \frac{a\cos\phi}{2}}{\rho_+^2} - \frac{\rho + \frac{a\cos\phi}{2}}{\rho_-^2}, \\ H_\phi &= \frac{a\rho(\rho^2 + \frac{a^2}{4})\sin\phi}{\rho_+^2\rho_-^2}, \qquad H_z = 0, \\ H &= \sqrt{H_\rho^2 + \frac{H_\phi^2}{\rho^2}} = \frac{a}{\rho_+\rho_-}. \end{aligned} \tag{83}$$

The corresponding vector potential is given by

$$\tilde{C}_\mu = zH_\phi\partial_\mu\rho - \rho zH_\rho\partial_\mu\phi. \tag{84}$$

The eigenvalue equation for the operator K has the form

$$\begin{aligned} \Big\{ -\frac{1}{\rho}\partial_\rho(\rho\partial_\rho) - \frac{1}{\rho^2}\partial_\phi^2 - \partial_z^2 - 2i\frac{z}{\rho}\Big(H_\phi\partial_\rho - H_\rho\partial_\phi\Big) \\ + z^2H^2 \pm 2H \Big\}\Psi(\rho,\phi,z) = E\Psi(\rho,\phi,z). \end{aligned} \tag{85}$$

The equation can be interpreated as a Schroedinger equation for a massless gluon in the magnetic field of monopole and anti-monopole string pair.

Let us analyse the equation qualitatively first. We will concentrate on the potentially dangerous $-2H$ potential in (85). The singularities of the term z^2H^2 determine the essential singularitites of the differential equation. One can try to extract the leading factor of the solution and look for a finite solution for the ground state in the form

$$\Psi(\rho,\phi,z) = (\pi\rho_+\rho_-)^{-\frac{1}{4}} \exp\big(-\frac{z^2}{2\rho_+\rho_-}\big)F(\rho,\phi), \quad (86)$$

where $F(\rho,\phi)$ is normalized

$$\int |F(\rho,\phi)|^2 \rho d\rho d\phi = 1. \quad (87)$$

The solution describes a wave function localized mainly near the string pair. The wave function vanishes exactly on the axes of the strings. This implies that the ground state has a non-zero orbital angular momentum which provides a centrifugal potential as we will see later.

The lowest negative eigenvalue of this equation can be obtained by variational method by minimizing the corresponding energy functional

$$\begin{aligned} E = \int \Psi^* \Big[-\frac{1}{\rho}\partial_\rho(\rho\partial_\rho) - \frac{1}{\rho^2}\partial_\phi^2 - \partial_z^2 - 2i\frac{z}{\rho}(H_\phi\partial_\rho \\ -H_\rho\partial_\phi) + z^2H^2 \pm 2H\Big]\Psi \rho d\rho dz d\phi. \quad (88) \end{aligned}$$

Now, with

$$F(\rho,\phi) = \sum_{-\infty}^{+\infty} f_m(\rho)\Phi_m, \quad (89)$$

one may suppose that the main contribution to the ground state energy comes from the first term of Fourier expansion with $m = 0$. With this we can perform the integration over z-coordinate, and simplify the above expression to

$$\begin{aligned} E = \int f(\rho)\Big(-f''(\rho) - \frac{1}{\rho}f'(\rho) \\ +U(\rho,\phi)f(\rho)\Big)\rho d\rho d\phi, \\ U(\rho,\phi) = \frac{\rho^2 - 2a\rho_+\rho_-}{2\rho_+^2\rho_-^2}, \quad (90) \end{aligned}$$

where $f(\rho) = f_0(\rho)$ and $U(\rho,\phi)$ is an effective potential. To understand the nature of the potential we make the following rescaling

$$\begin{aligned} \rho \to a\rho, \qquad f \to f/a, \\ E \to E/a^2, \quad (91) \end{aligned}$$

and find that under this rescaling the potential near the origin can be approximated to

$$U(\rho,\phi) \to -4a + (8 - 16a\cos 2\phi)\rho^2. \quad (92)$$

So after the rescaling the potential reduces to a two dimensional harmonic oscillator potential whose depth decreases as a goes to zero. This implies that the negative energy eigenvalues will disappear for a less than a finite critical value.

To complete our analysis we perform the integration over the angle variable ϕ in the energy functional, and with the change of variable

$$f(\rho) = \chi(\rho)/\sqrt{\rho}, \quad (93)$$

we obtain the following equation which minimizes the energy,

$$\begin{aligned} \Big[-\frac{d^2}{d\rho^2} + V(\rho)\Big]\chi(\rho) = E\chi(\rho), \\ V(\rho) = -\frac{1}{4\rho^2} + \frac{8\rho^2}{\sqrt{(a^4-16\rho^4)^2}} \\ -\frac{8a}{\pi\sqrt{(a^2-4\rho^2)^2}}K\big(-\frac{16a^2\rho^2}{(a^2-4\rho^2)^2}\big), \quad (94) \end{aligned}$$

where $K(x)$ is the complete elliptic integral of the first kind. With this we have the asymptotic behavior of the potential $V(\rho)$ at space infinity

$$V(\rho) \simeq (\frac{1}{4} - a)\frac{1}{\rho^2}. \quad (95)$$

This tells that the potential becomes positive when the distance a becomes less than the critical value a_0 (in the unit $1/\alpha$)

$$a < a_0 = \frac{1}{4}. \quad (96)$$

A careful numerical analysis of (85) gives us the following critical value

$$a_0 \simeq 0.24, \quad (97)$$

which is close to the approximate value 1/4. This confirms that qualitatively the approximate solution describes the correct physical picture. In particular, this tells that a pair of monopole and antimonopole strings become a stable magnetic background if the distance between two strings is small enough.

VIII. INSTABILITY OF MAGNETIC VORTEX-ANTIVORTEX PAIR

Recently an alternative mechanism of confinement has been proposed which advocates the condensation of magnetic vortices [26]. However, it has been known that the magnetic vortex configuration is unstable [27]. So it would be interesting to study the stability of the vortex-antivortex pair. In this section we study the stability of

the vortex and anti-vortex pair and show that the pair is unstable.

Let us start with a single vortex configuration given by

$$\vec{H} = \frac{1}{\rho}\hat{z}, \qquad \tilde{C}_\mu = \rho\partial_\mu\phi. \tag{98}$$

Notice that, unlike the monopole string (59), the vortex configuration is not a classical solution of the system. But since this type of configuration multiplied by an appropriate profile function has been studied by many authors [26], we will consider the vortex configuration in the following.

The corresponding eigenvalue equation of the operator K is given by

$$\Big[-\partial_\rho^2 - \frac{1}{\rho}\partial_\rho - \frac{1}{\rho^2}\partial_\phi^2 - \partial_z^2 - \frac{2i}{\rho}\partial_\phi \pm 2H\Big]\Psi(\rho,\phi,z) \\ = E\Psi(\rho,\phi,z). \tag{99}$$

The equation becomes separable in all three variables. With

$$\Psi = \sum_{-\infty}^{+\infty} f_m(\rho)g(z)\Phi_m(\phi), \qquad g(z) = 1, \tag{100}$$

one obtains the following ordinary differential equation for $f(\rho)$ from (99),

$$\Big(-\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + (1+\frac{m}{\rho})^2 \pm \frac{2}{\rho} - E\Big)f(\rho) \\ = 0. \tag{101}$$

The bound states are possible for the potential $-2/\rho$ with non-positive integer m, in which case the corresponding solution is given by

$$\begin{gathered} f_{n,m}(\rho) = \rho^{|m|}e^{-\sqrt{1-E_{n,m}}}u_{n,m}(\rho), \\ u_{n,m}(\rho) = \sum_{k=0}^{n} a_k^{n,m}\rho^k, \\ a_{k+1}^{n,m} = \\ \frac{\sqrt{1-E_{n,m}}(2k+2|m|+1) - 2 + 2m}{(k+1)(k+2|m|+1)}a_k^{n,m}, \\ E_{n,m} = 1 - \frac{4(1-m)^2}{(2n+2|m|+1)^2}, \\ n = 0,1,2,...;\ m = 0,-1,-2,... \end{gathered} \tag{102}$$

Clearly the ground state has a negative energy $E_{0,0}$, which tells that the vortex configuration is unstable.

There are two principal differences between the vortex configuration and the axially symmetric monopole string. First, the monopole string is a classical solution of $SU(2)$ QCD, but the vortex configuration is not. Secondly, the ground state eigenfunction $f_{0,0}(\rho)$ of the vortex configuration corresponds to an S-state, which implies the absence of the centrifugal potential. But the ground state of the monopole string has a non-trivial centrifugal potential. As we will see soon this will play the important role in the existence of the negative energy eigenstates in the case of vortex-antivortex background.

The vortex-antivortex background is described in a similar manner as the monopole-antimonopole string background in the last section. The potential is given by

$$\begin{aligned} \tilde{C}_\mu &= \frac{a}{2}\sin\phi(\frac{1}{\rho_+} + \frac{1}{\rho_-})\partial_\mu\rho \\ &+\Big[\frac{\rho}{\rho_+}(\rho + \frac{a}{2}\cos\phi) - \frac{\rho}{\rho_-}(\rho - \frac{a}{2}\cos\phi)\Big]\partial_\mu\phi, \\ \vec{H} &= \Big(\frac{1}{\rho_+} - \frac{1}{\rho_-}\Big)\hat{z}, \end{aligned} \tag{103}$$

where a is the distance between the axes of the vortex and anti-vortex. The eigenvalue equation corresponding to the operator K is given by

$$\Big[-\partial_\rho^2 - \frac{1}{\rho}\partial_\rho - \frac{1}{\rho^2}\partial_\phi^2 - \partial_z^2 - 2i(\tilde{C}_\rho\partial_\rho + \frac{1}{\rho^2}\tilde{C}_\phi\partial_\phi) \\ +\tilde{C}_\mu^2 \pm 2H\Big]F(\rho,\phi) = EF(\rho,\phi), \tag{104}$$

which is partially factorizable in z-coordinate. The numerical analysis of the equation shows that there is no critical value for the parameter a, so that the negative energy eigenvalues exist for any small a. Qualitatively one can see this from the effective the potential $\tilde{C}_\mu^2 - 2H$. After averaging over the angle variable the potential becomes

$$\begin{aligned} V(\rho) &= 2\int_0^{2\pi}(1 - |\frac{1}{\rho_+} - \frac{1}{\rho_-}| - \frac{\rho^2 - \frac{a^2}{4}}{\rho_+\rho_-})d\phi \\ &= 4\pi + \frac{16}{\rho+\frac{a}{2}}F(\frac{\pi}{4}, x_-) - \frac{16}{|\rho-\frac{a}{2}|}F(\frac{\pi}{4}, -x_-) \\ &\quad -\frac{8}{\rho+\frac{a}{2}}K(x_+) + \frac{8}{|\rho-\frac{a}{2}|}K(-x_-) \\ &\quad -\frac{8(\rho-\frac{a}{2})}{|\rho-\frac{a}{2}|}K(-\frac{x_+x_-}{4}), \\ x_\pm &= \frac{2a\rho}{(\rho\pm\frac{a}{2})^2} \end{aligned} \tag{105}$$

where $F(w,x), K(x)$ are the elliptic and the complete elliptic integral of the first kind. So near the origin and infinity we have

$$V(\rho) \simeq \begin{cases} 8\pi - \dfrac{64}{a^2}\rho - \dfrac{16\pi}{a^2}\rho^2 & (\rho \simeq 0), \\ -\big(4 - \dfrac{\pi}{2}a\big)\dfrac{2a}{\rho^2} & (\rho \simeq \infty). \end{cases} \tag{106}$$

This shows that the effective potential has no centrifugal potential which could prevent the appearance of bound states for small a. This is the origin of existence of the negative energy eigenvalues and instability of the vortex-antivortex background. Whether the instability problem can be overcome with a more complicate configuration of the vortex-antivortex is an open and interesting question.

IX. CONCLUSIONS

In this paper we have shown that the axially symmetric monopole-antimonopole string background is stable under the quantum fluctuation, if the distance between two string becomes less than the critical value $a_0 \simeq 1/4$. As far as we understand it this is the first explicit example of a stable magnetic background in $SU(2)$ QCD. The existence of the stable magnetic background strongly implies that "a spagetti of gauge invariant monopole-antimonopole string pairs" could generate a stable vacuum condensation in QCD. This would allow a magnetic confinement of color in QCD.

Another important result of this paper is that a pair of magnetic vortex and anti-vortex strings is unstable. In the magnetic confinement mechanism there have been two competing ideas, the monopole-antimonopole condensation and the magnetic vortex-antivortex condensation, and recently the magnetic vortex-antivortex condensation has been advocated by many authors as a possible confinement mechanism in QCD [26]. Our result suggests that the magnetic vortex-antivortex condensation is not likely to generate a stable magnetic vacuum.

The search for the existence of a stable magnetic condensation in QCD has been painful. Savvidy first calculated the one-loop effective action of $SU(2)$ QCD with a constant magnetic background. But the Savvidy background was unstable, because it was not gauge invariant. In fact any classical background which is not gauge invariant can not be physical, and thus can not be a stable vacuum. Because of this Nielsen and Olesen have proposed the gauge invariant "Copenhagen vacuum", and conjectured that such a gauge invariant vacuum must be stable under the quantum fluctuation [6]. Although conceptually very attractive the "Copenhagen vacuum", however, was not so useful in practical purposes.

But recently it has been shown that if we impose the gauge invariance to the SNO vacuum, the imaginary part of the effective action disappears [11]. This tells that if we imposes the gauge invariance properly, QCD can generates a stable magnetic condensation. The result in this paper strongly endorses this. Although the axially symmetric monopole-antimonopole string background can not be identified as a vacuum because it is not translationally invariant, it clearly indicates the existence of a stable magnetic vacuum. Furthermore, it indicates that a stable magnetic vacuum must be, not just a condensation of monopoles but, a condensation of a gauge invariant combination of monopoles and anti-monopoles. In fact our result strongly suggests that a "spagetti of monopole-antimonopole string pairs", or more precisely an infinite set of monopole-antimonopole string pairs which forms a square crystal in xy-plane, can be a stable QCD vacuum if the size of the crystal is small enough.

Finally we wish to point out a possible connection between our monopole-antimonopole string background and the "Copenhagen vacuum". The "Copenhagen vacuum" can be viewed as a collection of domains of constant magnetic fields which have random directions. Our result suggests that an infinite set of monopole-antimonopole string pairs which forms a square crystal in xy-plane can be viewed as a stable vacuum, and thus a candidate of "Copenhagen vacuum". In this sense our result strongly supports the idea of "Copenhagen vacuum". On the other hand, our result also imposes a strong constraint on the "Copenhagen vacuum". In a simplified picture the "Copenhagen vacuum" was proposed as a "spagetti of colored magnetic tubes" [6]. But our result shows that the magnetic vortex-antivortex pair is unstable. This implies that "spagetti of colored magnetic tubes" may not be viewed as a "Copenhagen vacuum". Rather, a "spagetti of monopole-antimonopole string pairs" forms a stable magnetic vacuum, and thus can be viewed as a "Copenhagen vacuum". This is consistent with the picture of "gauge invariant" monopole condensation.

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